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Multi-instanton calculus and equivariant cohomology

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Abstract: We present a systematic derivation of multi-instanton amplitudes in terms of ADHM equivariant cohomology. The results rely on a supersymmetric formulation of the localization formula for equivariant forms. We examine the cases of $\mathcal{N} = 4$ and $\mathcal{N} = 2$ gauge theories with adjoint and fundamental matter.

Keywords: Topological Field Theories, Nonperturbative Effects, BRST Symmetry.

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1. Introduction

This year has seen dramatic improvements in our capabilities to handle multi instanton calculus. This is because powerful localization methods have been applied to these computations.

The use of supersymmetric field theories to compute topological invariants was advocated in [1] and since then much work has been carried out to clarify the formalism and its applications. In our opinion, all of this work has not cleared a widespread belief that considered these methods not very relevant for "physical" cases. This paper is about one of such cases: the computation via path integral methods of non perturbative contributions due to instantons for Yang-Mills gauge theories with $\mathcal{N}=2,2^*,4$ supersymmetries (SYM).

The computation of non perturbative effects has been the focus of much recent research. Very often one studies such effects in the framework of string theory or supergravity and, as a by product, recovers SYM with a low energy limit. Non perturbative results for SYM thus provide important checks on these constructions like in the case of the AdS/CFT correspondence.

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Notwithstanding its importance, the problem of extracting non perturbative results from path integral computations has been not so intensively studied may be because every little advance has been at the cost of a lot of effort. A short "historical" *excursus* will clarify better what we mean.

To a "primordial" era in which the basic techniques were estabilished [2], aiming at supersymmetry breaking in gauge theories, it followed a period of stasis which was broken by the analysis of $\mathcal{N} = 2$ SYM carried out in [3]. To check this analysis, a lot of effort went in the computation of instanton effects for winding numbers, k, larger than one [4, 5]. Unfortunately these efforts were frustrated by the fact that the ADHM constraints can be solved in an explicit way only for $k = 1, 2$. Explicit results could then be obtained only for the above mentioned values of k. At this point it is worth to mention that in order to compare with the results obtained for the prepotential in [3] there are two main directions: the first one employed in [4] relied on the computation of correlators which can be extracted directly from the effective lagrangian and that exhibit a dependence on the space-time coordinates. The other, followed in [5], was to compute the correlator $u = \langle \mathrm{Tr}\, \varphi^2 \rangle$ where φ is the complex scalar field of the $\mathcal{N} = 2$ supersymmetric multiplet and u is the gauge invariant coordinate which describes the auxiliary Seiberg-Witten curve. Due to supersymmetric Ward identities, $\langle \mathrm{Tr}\, \varphi^2 \rangle$ is a pure number times a scale to the power of the naive dimension of the operator computed. The prepotential is then recovered using the identities of [6]. The advantage of the second approach, in the light of the developments that have happened since then, is that the computation of the correlator reduce to that of the partition function of a suitable matrix model defined on the moduli space of instantons, whose action is generated by the presence of a vev for the scalar field.

Since those early efforts and the most recent developments, there has been two main advances: on the one hand a measure for the moduli space of the instantons has been proposed which has led to many interesting results and that allows to write correlators without explicitly solving the constraints (for a complete review see [7]). On the other hand the agreement between the results of [4, 5], which were obtained in the semiclassical expansion, and those of [3] which did not have this shortcoming, triggered a subset of the present authors to recompute the correlators formulating the problem in the light of topological theories for which the semiclassical expansion is exact. In [8] this program was carried out: the action, after the imposition of the constraint, was written as the BRST variation of a suitable expression. The BRST charge thus defined squared to zero after properly taking into account all the symmetries of the theory (see later for more details). The first part of this program was completed in [9] in which the measure was shown to be BRST closed. The multi-instanton action, meausure and the entire correlator of interest can then be written in a BRST exact form as it was shown in [10] for $\mathcal{N} = 4$ and in [11] for $\mathcal{N} = 2$.

We now come to the last part of our story. Already in [8] the use of localization formulae was advocated in the study of multi instanton calculus. But at that time it was not clear to the authors how to deal with the singularities of the instanton moduli space and with its boundaries. In [8], in fact, a formula was given for the correlators as boundary terms over the moduli spaces of SU(2) instantons but explicit computations could be carried out only in the $k = 1$ case.

In [10] it appeared the proposal to deform the moduli space by minimally resolving the singularity using the invariance, under certain deformations, of the original BRST exact theory. Moreover, in the same paper, it was also suggested that the action had its minima in certain points that could be interpreted as resolved Hilbert schemes [12].[1] These ideas were then coherently applied in [13] in which the author recomputed the $k = 1, 2$ cases showing the full applicability of localization techniques to this problem.

There was a last crucial ingredient which was missing and it was provided in [14]: localization techniques are most powerful when the critical points of the action are isolated. In order to have such isolated points the BRST transformations of the theory must be further deformed by a rotation in the space-time which, in turn, induces an action on the moduli space. This action leaves the ADHM constraints invariant. The cohomology of the BRST operator is thus the same of that of the undeformed theory and it can be used safely if we find it to be more convenient. This technique has been used widely in the mathematical literature. It has also appeared in the physics literature, his most notable applications being the computation of the D-instanton partion function [15] and the study of the role of momentum maps for supersymmetric theories [16, 17]. This stream of physics literature might be traced back to the investigations of the possible use of the Duistermaat-Heckman formula in the context of supersymmetric gauge theories in the two dimensional case [19]. This will be a particular case of the most general situation we are going to treat in the next section.

At last we come to the content of this paper: we share with [20, 21], the belief that the ideal setting for these computations is that of equivariant cohomology (see [22] for a complete review). In the $\mathcal{N} = 2$ case the formulae found in [22] give the correct result because in this case the number of fermions and bosons are the same, and the fermions belongs to the tangent space of the bosonic moduli. There is also a way to avoid treating the constraints, by modifying some computations in [12]; but in the general case these formulae have to be modified. This is what we do here, by discussing the results that can be obtained from a proper supersymmetric formulation of the localization formulae for equivariant forms along the lines of [9]. Full details on this extension will appear elsewhere [23].

This is the plan of the paper: in the next section we carefully define and explain the objects which will enter the localization formula in the case of the bosonic theory. We keep the discussion as elementary as possible giving examples each time we introduce new objects. In the third and last section we first identify the objects introduced in the previous section as the building blocks of supersymmetric gauge theories. We then compute the $\mathcal{N} = 2, 2^*, 4$ cases. For the sake of clarity we have decided to relegate mathematical considerations and comparisons with previous literature to the appendices.

2. Preliminaries

2.1 The ADHM data

The moduli space of self-dual solutions of $\mathrm{U}(N)$ YM-equations in four dimensions is elegantly described by a $4kN$-dimensional hypersurface embedded in a $4kN + 4k^2$-dimensional

[1] Sometimes, in some physics literature, they are called improperly "U(1) instantons". We prefer the definition appearing in the mathematical literature.

ambient space via ADHM constraints. In the case of supersymmetric gauge theories the ADHM data are supplemented with fermionic moduli associated to zero modes of the gaugino field. The multi-instanton action is defined by plugging in the SYM lagrangian the bosonic and fermionic zero modes in terms of ADHM moduli and imposing the ADHM constraints via lagrangian multipliers.

There is a quick way to perform all of these steps at once and it is to use the corresponding D-brane description. The ADHM action for $\mathcal{N}=4$ can be extracted from the low energy dynamics of a system of k D(-1) and N D3-branes moving in flat space [28]–[31], the moduli of the four dimensional supersymmetric theory being the massless excitations of the open strings stretching between various branes. The complete ADHM lagrangian can, in fact, also be derived from the computation of disks amplitudes in string theory [26, 27].

This results into a zero-dimensional quantum theory of matrices, some transforming in the adjoint of U(k) and others in the bifundamental of U$(N)\times$U(k). Less supersymmetric multi-instanton actions are then found via suitable projections of the original $\mathcal{N}=4$ theory (see below for details).

Let us start by describing the $\mathcal{N}=4$ ADHM data [10, 25]. The position of k D(-1)-instantons in ten-dimensional space can be described by five complex fields B_ℓ,ϕ with $\ell=1,..4$. For latter convenience we have distinguished one of the complex planes and denoted it by ϕ. In addition open strings stretching between D(-1)-D3 branes provide two extra complex moduli I,J in the $(\bar{k},N)$ and $(\bar{N},k)$ bifundamental representations respectively of U$(k)\times$U(N). The U(N) group, together with the SO(4) $\times$ SO(6) Lorentz group preserved by the D(-1)-D3 system, act as the group of global isometries of the U(k) zero-dimensional quantum theory living in the D(-1)-worldvolume. Supersymmetry requires bosonic moduli to be paired with fermionic ones. Bosonic ADHM moduli in the adjoint of U(k) come together with sixteen fermionic components $(\chi_v,\eta,\mathcal{M}_\ell,\bar{\mathcal{M}}_\ell)$, $v=1,\ldots,7$, $\ell=1\ldots,4$, coming from the reduction under a subgroup SO(2) $\times$ SO(7) [32] of a single Majorana-Weyl fermion in $D=10$ down to zero dimensions. Again the splitting of the fermionic fields in $7+1$ real and 4 complex components is a matter of convenience.

On the other hand fermionic excitations of D(-1)-D3 open strings provide two pairs (μ_I,μ_K), (μ_J,μ_L) of complex fields in the $(\bar{k},N)$ and $(\bar{N},k)$ bifundamental representations respectively.

The last ingredient in the construction of the instanton moduli space is the ADHM constraints. ADHM constraints can be efficiently implemented by lagrangian multipliers. For this purpose it is convenient to introduce the auxiliary fields $K,L,H_{\mathbb{R}},H_r$ with $r=1,2,3$. To the adjoint auxiliary fields $H_{\mathbb{R}},H_r$ we associate respectively one real and three complex functions

$$\begin{aligned}
\mathcal{E}^{\text{adj}}_{\mathbb{R}} &= \left[B_\ell,B_\ell^\dagger\right]+II^\dagger-J^\dagger J-\zeta\,,\\
\mathcal{E}^{\text{adj}}_{1} &= [B_1,B_2]+\left[B_3^\dagger,B_4^\dagger\right]+IJ\,,\\
\mathcal{E}^{\text{adj}}_{2} &= [B_1,B_3]-\left[B_2^\dagger,B_4^\dagger\right]\,,\\
\mathcal{E}^{\text{adj}}_{3} &= [B_1,B_4]+\left[B_2^\dagger,B_3^\dagger\right]\,,
\end{aligned}\tag{2.1}$$

and to the fundamental auxiliary fields (K, L) the two complex functions

$$\begin{aligned}\mathcal{E}_K^{\rm fun} &= B_3 I - B_4^\dagger J^\dagger \\ \mathcal{E}_{L^\dagger}^{\rm fun} &= B_4 I + B_3^\dagger J^\dagger\,.\end{aligned} \tag{2.2}$$

As we will see in the next section, after reduction to $\mathcal{N}=2$, (2.1) will reduce to the familiar ADHM constraints. For greater generality we have added a non commutativity parameter ζ to minimally resolve the small instantons singularities of the moduli space. In the following we will collect the 3 complex and 1 real adjoint components in (2.1) in the seven-vector $\mathcal{E}_v^{\rm adj}$, $v = 1, \ldots, 7$, and denote the associated auxiliary fields as χ_v, H_v.

The ADHM data just described can be nicely organized in multiplets of a BRST current Q [10].[2] Q is a BRST current in the sense that it squares to zero up to a $\mathrm{U}(k)$ gauge transformation on the moduli space. We will need a slight modification of this BRST charge, in which the $\mathrm{U}(k)$ group action is combined with an element of a $\mathrm{U}(1)^{N-1}\times\mathrm{U}(1)^3$ subgroup in the $\mathrm{SU}(N)\times\mathrm{SO}(4)\times\mathrm{SO}(6)$ global symmetry group of the D(-1)-D3 system. This choice is dictated by the requirement that the BRST charge closes up to a group action with isolated critical points. As firstly appreciated in [14] this allows to reduce integrals in the ADHM moduli space to a sum over critical points. In the next subsection we will state our main localization formula for group actions meeting this basic requirement. The reader interested in a deeper understanding of the discussion in this section and the connection with previous results in [8, 25] is referred to the appendix B.

We denote the new BRST charge by Q_ϵ and parametrize an element in $\mathcal{T} = \mathrm{U}(1)^{N-1}\times \mathrm{U}(1)^3$ by $a_\lambda, \epsilon_1, \epsilon_2, m$ with $\lambda = 1, \ldots, N$ and $\sum a_\lambda = 0$. The $\mathrm{U}(1)_{\epsilon_{1,2}}$'s are inside the $\mathrm{SO}(4)$ Lorentz group while $\mathrm{U}(1)_m$ is chosen in $\mathrm{SO}(6)$. The m-deformation breaks the $\mathrm{SO}(6)\sim\mathrm{SU}(4)$ $\mathcal{R}$-symmetry group of $\mathcal{N}=4$ down to the $\mathrm{SU}(2)\times\mathrm{U}(1)$ $\mathcal{R}$-symmetry group of $\mathcal{N}=2$. The $\mathcal{N}=4$ adjoint vector multiplet decomposes into a vector and an hypermultiplet of $\mathcal{N}=2$. Later we will identify the parameter m with the mass of the hypermultiplet. Keeping this identification in mind we call the deformed $\mathcal{N}=4$ theory as $\mathcal{N}=2^*$. Pure $\mathcal{N}=2$ SYM theory can instead be defined by a $\mathbb{Z}_2$-projection with $\mathbb{Z}_2\subset\mathrm{U}(1)_m$.

Given all of this, the deformed $\mathcal{N}=2^*$ multi-instanton action can be written as the Q_ϵ-exact form [10]:

$$S^{\mathcal{N}=2^*} = Q_\epsilon\,\mathrm{Tr}\left[\frac{1}{4}\eta\,[\phi,\bar\phi] + \vec{H}\cdot\vec\chi - i\vec{\mathcal{E}}\cdot\vec\chi - \frac{1}{2}\sum_{s=1}^{6}\left(\Psi_s^\dagger(\bar\phi+\lambda_s)\cdot B_s + \Psi_s(\bar\phi+\lambda_s)\cdot B_s^\dagger\right)\right] \tag{2.3}$$

with $B_s = (I, J^\dagger, B_\ell)$, $\Psi_s = (\mu_I, \mu_J^\dagger, \mathcal{M}_\ell)$, $\vec\chi = (\mu_K, \mu_{L^\dagger}, \chi_v)$, $\vec{H} = (K, L^\dagger, H_v)$ and $\vec{\mathcal{E}} = (\mathcal{E}_K^{\rm fun}, \mathcal{E}_{L^\dagger}^{\rm fun}, \mathcal{E}_v^{\rm adj})$. The convention for the vector product is $\vec\chi\cdot\vec{H}\equiv\frac{1}{2}\chi_{\mathbb{R}}H_{\mathbb{R}} + \chi_r^\dagger H_r + \chi_r H_r^\dagger$, while $\phi\cdot B_s = [\phi, B_s]$ or $\phi\cdot B_s = \phi B_s$ depending or whether B_s is in the $\mathrm{U}(k)$ adjoint or fundamental representation respectively .

[2] See [8, 11] for the $\mathcal{N}=2$ case

The BRST transformations are given by:

$$
\begin{aligned}
Q_\epsilon I &= \mu_I \qquad Q_\epsilon \mu_I = \phi I - I a \\
Q_\epsilon J &= \mu_J \qquad Q_\epsilon \mu_J = -J\phi + aJ + \epsilon J \\
Q_\epsilon \mu_K &= K \qquad Q_\epsilon K = \phi\mu_K - \mu_K a - m\mu_K \\
Q_\epsilon \mu_L &= L \qquad Q_\epsilon L = -\mu_L \phi + a\mu_L + (\epsilon - m)\mu_L \,, \\
Q_\epsilon B_\ell &= \mathcal{M}_\ell \qquad Q_\epsilon \mathcal{M}_\ell = [\phi, B_\ell] + \lambda_\ell B_\ell \\
Q_\epsilon \chi_v &= H_v \qquad Q_\epsilon H_v = [\phi, \chi_v] + \lambda_v \chi_v \\
Q_\epsilon \bar{\phi} &= \eta \qquad Q_\epsilon \eta = [\phi, \bar{\phi}] \\
Q_\epsilon \phi &= 0 \,,
\end{aligned}
\tag{2.4}
$$

with $\epsilon = \epsilon_1 + \epsilon_2$, $\phi \in \mathrm{U}(k)$ and

$$
\begin{aligned}
a &= \mathrm{diag}\left(e^{ia_1}, e^{ia_2}, \dots e^{ia_N}\right) \\
\lambda_s &= (\lambda_I, \lambda_{J^\dagger}; \lambda_\ell) = (0, -\epsilon; \epsilon_1, \epsilon_2, -m, m-\epsilon) \\
\lambda_v &= (\lambda_{\mathbb{R}}; \lambda_r) = (0; \epsilon, \epsilon_1 - m, m - \epsilon_2) \,.
\end{aligned}
\tag{2.5}
$$

The group assigments in the right-hand side of (2.4) are determined by the requirement that auxiliary fields associated to (2.1) transform covariantly under the $\mathrm{U}(k) \times \mathrm{SU}(N) \times \mathrm{U}(1)^3$ transformations:

$$
\begin{aligned}
I &\to g_{\mathrm{U}(k)}\, I\, g^\dagger_{\mathrm{U}(N)} \\
J &\to e^{i\epsilon}\, g_{\mathrm{U}(N)}\, J\, g^\dagger_{\mathrm{U}(k)} \\
K &\to e^{-im}\, g_{\mathrm{U}(k)}\, K\, g^\dagger_{\mathrm{U}(N)} \\
L &\to e^{i(m-\epsilon)}\, g_{\mathrm{U}(N)}\, L\, g^\dagger_{\mathrm{U}(k)} \\
B_\ell &\to e^{i\lambda_\ell}\, g_{\mathrm{U}(k)}\, B_\ell\, g^\dagger_{\mathrm{U}(k)} \\
H_v &\to e^{i\lambda_v}\, g_{\mathrm{U}(k)}\, H_v\, g^\dagger_{\mathrm{U}(k)}
\end{aligned}
\tag{2.6}
$$

with λ_ℓ, λ_v given by (2.5) and similar expressions for the fermionic superpartners.

2.2 Equivariant forms and the localization formula

Let M be an n-dimensional manifold acted on by a Lie group G with Lie algebra $\mathfrak{g}$. For every $\xi \in \mathfrak{g}$ we denote by ξ^* the *fundamental vector field* associated with ξ, i.e., the vector field that generates the one-parameter group $e^{t\xi}$ of transformations of M. Locally one has

$$
\xi^* = \xi^\alpha\, T^i_\alpha \frac{\partial}{\partial x^i}
$$

where the ξ^α are the components of ξ in some chosen basis of $\mathfrak{g}$, and the T^i_α are functions (the generators of the action).

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Let $\alpha : \mathfrak{g} \to \Omega(M)$ a polynomial map from $\mathfrak{g}$ to the algebra of differential forms on M. α may be regarded as an element of $\mathbb{C}[\mathfrak{g}] \otimes \Omega(M)$ with $\mathbb{C}[\mathfrak{g}]$ the algebra of complex-valued polynomials on $\mathfrak{g}$. We define a grading in $\mathbb{C}[\mathfrak{g}] \otimes \Omega(M)$ by letting, for homogeneous $P \in \mathbb{C}[\mathfrak{g}]$ and $\beta \in \Omega(M)$,

$$\deg(P \otimes \beta) = 2\deg(P) + \deg(\beta) . \tag{2.7}$$

The action of the group G on an element $\alpha \in \mathbb{C}[\mathfrak{g}] \otimes \Omega(M)$ is defined to be

$$(g \cdot \alpha)(\xi) = g^*(\alpha(Ad_{g^{-1}}\xi)) \tag{2.8}$$

where g^* denotes the pullback of forms with respect to the map $g\colon M \to M$. Elements α such that $g \cdot \alpha = \alpha$ are called G-equivariant forms. The equivariant differential $\mathcal{D}$ is defined by letting

$$(\mathcal{D}\alpha)(\xi) = d(\alpha(\xi)) - i_{\xi^*}\alpha(\xi) \tag{2.9}$$

where i_{ξ^*} is the inner product by the vector field ξ^*.

As an example let us take $G = O(2)$, $M = \mathbb{R}^2 - \{0\}$ with the standard action of G, and

$$\alpha(\xi) = \xi r d\theta$$

(we write the matrices in the Lie algebra $\mathfrak{so}(2)$ as $\left(\begin{smallmatrix} 0 & \xi \\ -\xi & 0 \end{smallmatrix}\right)$). Explicit computation shows that α is equivariant. Since

$$\xi^* = 2\xi \frac{\partial}{\partial\theta} \tag{2.10}$$

one has

$$(\mathcal{D}\alpha)(\xi) = d(\alpha(\xi)) - i_{\xi^*}\alpha(\xi) = -\xi dr \wedge d\theta - \xi r^2 . \tag{2.11}$$

In the first term on the r.h.s. of (2.11) the degree of the one-form $\alpha(\xi)$ has been raised by one unit, while in the second term it has been lowered by one unit. But in this very term the degree of the polynomial in $\mathbb{C}[\mathfrak{g}]$ has been raised by one and therefore according to (2.7) the total degree is raised by one unit, as we expect from a derivation.

Acting twice on a G-equivariant form α one finds:

$$\left(\mathcal{D}^2\alpha\right)(\xi) = (d - i_{\xi^*})^2\alpha(\xi) = -(di_{\xi^*} + i_{\xi^*}d)\alpha(\xi) = -\mathcal{L}_{\xi^*}\alpha(\xi) = 0 \tag{2.12}$$

since α is equivariant. The space of equivariant differential forms with this differential, graded with the degree (2.7), is a differential complex; its cohomology is called the G-equivariant cohomology of M.

We shall denoted by $\alpha_i(\xi)$ the homogeneous component of degree i of the differential form $\alpha(\xi)$. The condition that α is equivariantly closed, $\mathcal{D}\alpha = 0$, implies that $\alpha_n(\xi)$ (with $n = \dim M$) is exact outside of the set M_0 of zeros of ξ^* [22], suggesting that the integral $\int_M \alpha(\xi)$ reduces to an integral over M_0. This is the content of the localization formula below.

Let now x_0 be a zero of ξ^*. We introduce a map $\mathcal{L}_{x_0} : T_{x_0}M \to T_{x_0}M$ defined as

$$\mathcal{L}_{x_0}(v) = [\xi^*, v] = -\xi^\alpha v^i \left(\frac{\partial T^j_\alpha}{\partial x^i}\right)_{x_0} \frac{\partial}{\partial x^j} , \tag{2.13}$$

(which makes sense because at the critical points the components of the fundamental vector field vanish, $\xi^\alpha T^i_\alpha(x_0) = 0$).

In particular cases $\mathcal{L}_{x_0}$ can be interpreted as a Hessian; this is the case of Morse theory. But a Hessian is defined given a certain reference function to be derived twice. On the contrary $\mathcal{L}_{x_0}$ is known once we know the group action. The reader should keep this in mind since this is a fact which will be of great relevance in the next section.

Given all of this, assuming that both M, G are compact, α equivariantly closed, and that $\xi \in \mathfrak{g}$ is such that the vector field ξ^* has only isolated zeroes, we can state the localization theorem[3]

$$\int_M \alpha(\xi) = (-2\pi)^{n/2} \sum_{x_0} \frac{\alpha_0(\xi)(x_0)}{\det^{\frac{1}{2}} \mathcal{L}_{x_0}} . \tag{2.14}$$

By $\alpha_0(\xi)$ we mean the part of $\alpha(\xi)$ which is a zero-form.

There is a nice supersymmetric formulation of this equation. A complete proof of this result will be presented in [23], here we just state the result. Let $\mathfrak{M}$ be a (n, n)-dimensional supermanifold, defined in such a way that the superfunctions on $\mathfrak{M}$ are (non-homogeneous) differential forms. This condition in particular implies a splitting $T_{x_0}\mathfrak{M} = T_{x_0}M \oplus T_{x_0}M$, and one can introduce a (odd) linear transformation $\Pi\colon T_{x_0}\mathfrak{M} \to T_{x_0}\mathfrak{M}$ defined by exchanging the two copies of $T_{x_0}M$. If (x^i) is a local coordinate chart on M, and (θ^j) a local basis of differential 1-forms, then (x^i, θ^j) is a local coordinate chart on the supermanifold $\mathfrak{M}$.

The group G acts naturally on $\mathfrak{M}$, extending the action on M. For every $\xi \in \mathfrak{g}$ one has an even supervector field on $\mathfrak{M}$,

$$\hat{\xi}^* = \xi^\alpha\, T^i_\alpha \frac{\partial}{\partial x^i} + \xi^\alpha\, \theta^j \frac{\partial T^i_\alpha}{\partial x^j} \frac{\partial}{\partial \theta^i} . \tag{2.15}$$

One can also introduce an odd vector field Q^* on $\mathfrak{M}$, defined as

$$Q^* = \theta^i \frac{\partial}{\partial x^i} + \xi^\alpha\, T^i_\alpha \frac{\partial}{\partial \theta^i} . \tag{2.16}$$

A simple computation shows that *the anticommutator of Q^* with itself is twice the generator $\hat{\xi}^*$*:

$$\frac{1}{2}\{Q^*, Q^*\} = \hat{\xi}^* .$$

The vector field Q^* here may be regarded as the *infinitesimal generator of the BRST transformations*.

The localization formula can now be stated as follows

$$\int_M \alpha(\xi) = (-2\pi)^{\frac{n}{2}} \sum_{x_0} \frac{\alpha_0(\xi)(x_0)}{|\mathrm{S}\det'\, \mathcal{L}_{x_0}|^{\frac{1}{2}}} , \tag{2.17}$$

where now $\mathcal{L}_{x_0} : T_{x_0}\mathfrak{M} \to T_{x_0}\mathfrak{M}$ is given by $\mathcal{L}_{x_0} = [Q^*, v]$, and the operator Sdet$'$ is defined as Sdet $\circ\, \Pi$. With this provision, from now on each time we write the superdeterminant we intend it to be primed.

[3] The interested reader can consult [22] for a proof.

3. Applications to supersymmetric gauge theories

From (2.3) we conclude that the $\mathcal{N}=4$ multi-instanton action and their descendants upon mass deformation, $\mathcal{N}=2^*$, or orbifold projection, $\mathcal{N}=2$, are equivariantly exact forms. Together with the $\mathrm{U}(k)\times\mathrm{U}(N)\times\mathrm{U}(1)^3$-invariance this implies that they are equivariantly closed. We can thus apply the localization techniques explained in the previous section to compute the multi-instanton partition function for all these theories. The appendix B collect some background material that can help for a deeper understanding of the results presented in this section.

3.1 $\mathcal{N}=2$ supersymmetric theories with gauge group $\mathrm{SU}(N)$

We start discussing the case of $\mathcal{N}=2$ gauge theories with N_F fundamental hypermultiplets. Let us first recall the content of the $\mathcal{N}=2$ ADHM data in the pure $\mathcal{N}=2$ case $N_F=0$. Pure $\mathcal{N}=2$ SYM theory can be obtained by placing a stack of N fractional D3-branes at a $\mathbb{R}^4/\mathbb{Z}_2$ singularity. The ADHM instanton moduli space can then be described by implementing the $\mathbb{Z}_2$-projection on the D(-1)-D3 system describing the $\mathcal{N}=4$ parent theory [11]. The net effect of such operation is to break the $\mathrm{SU}(4)$ $\mathcal{R}$-symmetry group of the $\mathcal{N}=4$ theory down to $\mathrm{SU}(2)_{\dot{A}}\times\mathrm{U}(1)_{\mathcal{R}}$, where $\mathrm{SU}(2)_{\dot{A}}$ is the $\mathcal{N}=2$ automorphism group and $\mathrm{U}(1)_{\mathcal{R}}$ the anomalous $\mathcal{R}$ charge. Choosing a $\mathbb{Z}_2\subset\mathrm{U}(1)_m$, the projection corresponds to set to zero all the fields charged under $\mathrm{U}(1)_m$ in (2.6)

$$B_3=B_4=K=L=H_2=H_3=0 \tag{3.1}$$

together with their fermionic superpartners. The field content is thus reduced to $B_{\hat{s}}=(I,J^\dagger,B_{\hat{\ell}})$, $\Psi_{\hat{s}}=(\mu_I,\mu_J^\dagger,\mathcal{M}_{\hat{\ell}})$, with $\hat{s}=1,\ldots,4$, $\hat{\ell}=1,2$, and $\vec{\chi}=\chi_{\hat{v}}$ and $\vec{H}=H_{\hat{v}}$ with $\hat{v}=1,2,3$.

The $\mathcal{N}=2$ multi-instanton action is then obtained from (2.3) by simply replacing s,v with $\hat{s},\hat{v}$

$$S^{\mathcal{N}=2}=Q_\epsilon\,\mathrm{Tr}\left[\frac{1}{4}\eta\left[\phi,\bar{\phi}\right]+\vec{H}\cdot\vec{\chi}-i\vec{\mathcal{E}}\cdot\vec{\chi}-\frac{1}{2}\sum_{\hat{s}=1}^{4}\left(\Psi_{\hat{s}}^\dagger\left(\bar{\phi}+\lambda_{\hat{s}}\right)\cdot B_{\hat{s}}+\Psi_{\hat{s}}\left(\bar{\phi}+\lambda_{\hat{s}}\right)\cdot B_{\hat{s}}^\dagger\right)\right] \tag{3.2}$$

In particular the surviving equations in (2.1) reproduce the familiar ADHM constraints:

$$\begin{aligned}\mathcal{E}_{\mathbb{R}}&=\left[B_1,B_1^\dagger\right]+\left[B_2,B_2^\dagger\right]+II^\dagger-J^\dagger J-\zeta=0\\ \mathcal{E}_{\mathbb{C}}&=[B_1,B_2]+IJ=0\,.\end{aligned} \tag{3.3}$$

The presence in (3.3) of the non commutativity parameter ζ allows to minimally resolve the orbifold singularities of the moduli space. The action (3.2) is invariant under

$$\begin{aligned}Q_\epsilon I&=\mu_I & Q_\epsilon\mu_I&=\phi I-Ia\\ Q_\epsilon J&=\mu_J & Q_\epsilon\mu_J&=-J\phi+aJ+\epsilon J\\ Q_\epsilon B_{\hat{\ell}}&=\mathcal{M}_{\hat{\ell}} & Q_\epsilon\mathcal{M}_{\hat{\ell}}&=[\phi,B_{\hat{\ell}}]+\epsilon_{\hat{\ell}}B_{\hat{\ell}}\\ Q_\epsilon\chi_{\hat{v}}&=H_{\hat{v}} & Q_\epsilon H_{\hat{v}}&=[\phi,\chi_{\hat{v}}]+\lambda_{\hat{v}}\chi_{\hat{v}}\\ Q_\epsilon\bar{\phi}&=\eta & Q_\epsilon\eta&=\left[\phi,\bar{\phi}\right]\\ Q_\epsilon\phi&=0\,. \end{aligned} \tag{3.4}$$

with $\hat{\ell}=1,2$ and $\lambda_{\hat{v}}=(\lambda_{\mathbb{R}},\lambda_{\mathbb{C}})=(0,\epsilon)$.

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We would like now to apply the localization formula (2.17) to compute the partition function in the multi-instanton moduli space described by the bosonic $\mathcal{B} = (I, J, B_{\hat{\ell}}, H_{\hat{v}}, \bar{\phi})$ and fermionic $\mathcal{F} = (\mu_I, \mu_J, \mathcal{M}_{\hat{\ell}}, \chi_{\hat{v}}, \eta)$ variables.

To this end we start by introducing the vector field Q^* generating the BRST transformations on the supermanifold and discussing the critical points of its action. From (3.4) we get

$$
\begin{aligned}
Q^* &= \mu_I \frac{\partial}{\partial I} + \mu_J \frac{\partial}{\partial J} + \mathcal{M}_{\hat{\ell}} \frac{\partial}{\partial B_{\hat{\ell}}} + H_{\hat{v}} \frac{\partial}{\partial \chi_{\hat{v}}} + \eta \frac{\partial}{\partial \bar{\phi}} + ([\phi, \chi_{\hat{v}}] + \lambda_{\hat{v}} \chi_{\hat{v}}) \frac{\partial}{\partial H_{\hat{v}}} + \\
&\quad + [\phi, \bar{\phi}] \frac{\partial}{\partial \eta} + (\phi - a) I \frac{\partial}{\partial \mu_I} + (-\phi + a + \epsilon) J \frac{\partial}{\partial \mu_J} + ([\phi, B_{\hat{\ell}}] + \epsilon_{\hat{\ell}} B_{\hat{\ell}}) \frac{\partial}{\partial \mathcal{M}_{\hat{\ell}}} \\
&= (Q^*)^i_{\mathcal{B}} \frac{\partial}{\partial \mathcal{B}^i} + (Q^*)^i_{\mathcal{F}} \frac{\partial}{\partial \mathcal{F}^i},
\end{aligned} \tag{3.5}
$$

The critical points $Q^* = 0$ are given by setting to zero the components in (3.5)

$$
\begin{aligned}
(\varphi_{IJ} + \epsilon_\ell)\, B^\ell_{IJ} &= 0 \\
(\varphi_I - a_\lambda)\, I_{I\lambda} &= 0 \\
(-\varphi_I + a_\lambda + \epsilon)\, J_{\lambda I} &= 0
\end{aligned} \tag{3.6}
$$

with $H_{\hat{v}}$ and all fermions set to zero. $H_{\hat{v}} = 0$ implements the ADHM constraints (3.3). Equations (3.6) were solved in [14]. Each critical point is associated to a set of N Young Tableaux $(Y_1, \ldots Y_N)$ with $k = \sum_\lambda k_\lambda$ boxes distributed between the Y_λ's. The boxes in a Y_λ diagram are labeled either by the instanton index $I_\lambda = 1, \ldots, k_\lambda$ or by the pair of integers i_λ, j_λ denoting the vertical and horizontal position respectively in the Young diagram. We denote by $\nu_{i_\lambda}, \nu'_{j_\lambda}$ the length of the i_λ-th row and j_λ-th column respectively. The solution can then be written as:

$$
\varphi_{I_\lambda} = \varphi_{i_\lambda j_\lambda} = a_\lambda - (j_\lambda - 1)\epsilon_1 - (i_\lambda - 1)\epsilon_2 \tag{3.7}
$$

and $J = B_\ell = I = 0$ except for the components $B^1_{(i_\lambda j_\lambda + 1),(i_\lambda j_\lambda)}$, $B^2_{(i_\lambda + 1 j_\lambda),(i_\lambda j_\lambda)}$, $I_{\lambda,(i_\lambda = j_\lambda = 1)}$. These apparent moduli correspond to the zero eigenvalues in the left hand side of (3.6). They are however eliminated by the ADHM constraints (3.3). This can be seen as follows: at the critical points

$$
\mathcal{E}_{\mathbb{C}} = [B_1, B_2] = 0 \tag{3.8}
$$

is non-trivial only when the box labelled by the pair $(i_\lambda j_\lambda)$ has a neighbor both on its left and its down direction. These equations can then be used to determine, for example the corresponding components of B_2. This leaves $\nu_{1_\lambda} - 1$ undetermined components for B_2. In addition we have $k_\lambda - \nu_{1_\lambda}$ non-trivial B_1 components and one component for I. All together this leaves k components which are fixed by the diagonal components of the real constraint

$$
\mathcal{E}_{\mathbb{R}} = \left[B_1, B_1^\dagger\right] + \left[B_2, B_2^\dagger\right] + II^\dagger - \zeta = 0\,. \tag{3.9}
$$

We conclude that critical points of the $\mathrm{U}(1)^k \times \mathrm{U}(1)^{N-1} \times \mathrm{U}(1)^2$ action are isolated. We are now ready to apply the localization formula. As in [14] we can use the $\mathrm{U}(k)$-invariance to write the Q_ϵ-unpaired field ϕ as $\phi_{IJ} = \varphi_I - \varphi_J$ in terms of k φ_I phases. The jacobian of this change of variables brings the so called Vandermonde determinant $\prod_{I<J} \varphi_{IJ}^2$. According to our localization formula (2.17), we find

$$\mathcal{Z}_k = \int \frac{\mathcal{D}\phi}{\mathrm{U}(k)} \mathcal{D}\mathcal{B}\mathcal{D}\mathcal{F} e^{-S} = \int \prod_{I=1}^{k} d\varphi_I \frac{\prod_{I<J} \varphi_{IJ}^2}{\mathrm{S}\det \mathcal{L}} \equiv \sum_{x_0} \frac{1}{\mathrm{S}\det} \hat{\mathcal{L}}_{x_0}\,, \tag{3.10}$$

having used the fact that the action $S^{\mathcal{N}=2}$ vanishes on the critical points, thus $\alpha_0(\xi)(x_0) = 1$. The superdeterminant[4] is defined by

$$\mathrm{S}\det \mathcal{L} = S\det \begin{pmatrix} \frac{\partial (Q^*)^i_{\mathcal{B}}}{\partial \mathcal{F}^j} & \frac{\partial (Q^*)^i_{\mathcal{B}}}{\partial \mathcal{B}^j} \\ \frac{\partial (Q^*)^i_{\mathcal{F}}}{\partial \mathcal{F}^j} & \frac{\partial (Q^*)^i_{\mathcal{F}}}{\partial \mathcal{B}^j} \end{pmatrix}. \tag{3.11}$$

Plugging (3.11) in (3.10) one recovers (3.10) in [14] where the integral is computed in the complex plane with poles at the critical points (3.7). The explicit form of the residue formula obtained in this way is however difficult to handle. In the following we shall adopt the approach of [20][5] that generalize to $\mathrm{U}(N)$ the analysis performed by Nakajima [12] in the study of resolved Hilbert schemes. Following [20] we start by computing the character $\chi \equiv \sum_i (-)^F e^{i\lambda_i}$, where the λ_i's are the eigenvalues of $\mathcal{L}_{x_0}$ and $(-)^F = \pm 1$ according to the gradation given by (3.5).

As we will see the resulting character χ can be reduced via algebraic manipulations to a sum over $2kN$ eigenvalues. The determinant is then found by replacing the sum by a product over the $2kN$ eigenvalues.

Notice that the extension of the localization formula to the superspace allows to easily handle the linearized ADHM constraints by introducing the fermionic "ghost" variables $(\chi_{\mathbb{R}}, \chi_{\mathbb{C}})$. As we will shortly see, in the computation one can in fact nicely recognize the cancellations between bosonic and fermionic contributions that mimics the reduction via ADHM contraints to the $2kN$-dimensional moduli space (see appendix C for details). This is a general feature of the superspace approach, not necessarily linked to space-time supersymmetry.

Let us introduce the generators $T_\ell = e^{i\epsilon_\ell}, T_{a_\lambda} = e^{ia_\lambda}$ for elements in $\mathrm{U}(1)^{N-1} \times \mathrm{U}(1)^2$. In addition we write $V = e^{i\varphi_I}$ with φ_I given by (3.6). The Supertrace of $\hat{\mathcal{L}}_{x_0}$ at the critical point (3.6) can be written (see appendix C for a more detailed explanation) as

$$\chi = V^* \times V \times [T_1 + T_2 - T_1 T_2 - 1] + W^* \times V + V^* \times W \times T_1 T_2 \tag{3.12}$$

with

$$\begin{aligned} V &= \sum_{\lambda=1}^{N} \sum_{j_\lambda=1}^{\nu_{1_\lambda}} \sum_{i_\lambda=1}^{\nu'_{j_\lambda}} T_1^{-j_\lambda+1} T_2^{-i_\lambda+1} T_{a_\lambda} = \sum_{\lambda=1}^{N} \sum_{i_\lambda=1}^{\nu'_{1_\lambda}} \sum_{j_\lambda=1}^{\nu_{i_\lambda}} T_1^{-j_\lambda+1} T_2^{-i_\lambda+1} T_{a_\lambda}\,, \\ W &= \sum_{\lambda=1}^{n} T_{a_\lambda}\,. \end{aligned} \tag{3.13}$$

[4] Since our ADHM variables are complex, our tangent space is complex too and the determinant to the inverse of the square root becomes simply the inverse of the determinant in the localization formulae.

[5] We thank R. Flume and R. Poghossian for detailed explanations of their work.

The sum in V run over $I = 1, \dots, k$ distributed between the Young tableaux Y_λ's. The first three terms between brackets in (3.12) come from the U(k) adjoint fields $B_1, B_2, \chi_{\mathbb{C}}$ in (3.4). The -1 inside the bracket comes from the Vandermonde determinant. The last two terms are associated to the $I^\dagger$, $J^\dagger$ bifundamentals respectively. After a long but straight algebra (see [12] for details) one finds [21]:

$$\chi = \sum_{\lambda,\tilde{\lambda}}^{N} \sum_{s \in Y_j} \left(T_{a_{\lambda\tilde{\lambda}}} T_1^{-h(s)} T_2^{v(s)+1} + T_{a_{\tilde{\lambda}\lambda}} T_1^{h(s)+1} T_2^{-v(s)} \right) \tag{3.14}$$

with

$$h(s) = \nu_{i_\lambda} - j_\lambda \qquad v(s) = \tilde{\nu}'_{j_\lambda} - i_\lambda \,. \tag{3.15}$$

Notice that $\tilde{\nu}'_{j_\lambda}$ is defined only for $j_\lambda \leq \tilde{\nu}_{1_\lambda}$. For $j_\lambda > \tilde{\nu}_{1_\lambda}$ we take $\tilde{\nu}'_{j_\lambda} = 0$. $h(s)$ $(v(s))$ is the number of black (white) circles in figure1.

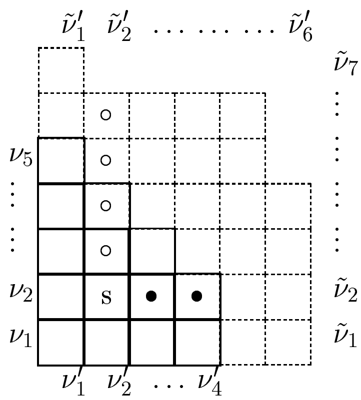

Figure 1: Two generic Young diagrams denoted by the indices $\lambda, \tilde{\lambda}$ in the main text.

The sum in (3.14) runs over $2kN$ eigenvalues, the complex dimension of the moduli space. Moreover for generic $a_\lambda, \epsilon_1, \epsilon_2$ there are no zero eigenvalues in (3.14), since for $\lambda = \tilde{\lambda}$, the quantities $h(s), v(s)$ are non-negative. Cancellations of zero eigenvalues in (3.12) can be traced to the term $V^* \times V \times (T_1 - 1)(1 - T_2)$. Zero eigenvalues are associated to the non-zero components of B_1, B_2, I which we have discussed above. Roughly the factor $(T_1 - 1)$ takes care of the cancellations connected to the B_1 components (horizontal neighbors) and $(1-T_2)$ does the same for B_2 (vertical neighbors) in the sum in $V^* \times V \times (T_1-1)(1-T_2)$. As anticipated, this mimics the reduction via ADHM constraints, since the negative contributions inside the brackets $-T_1T_2-1$ can be associated to the fermionic ADHM constraints implemented by $\chi_{\mathbb{C}}, \chi_{\mathbb{R}}$ and to the U(k) invariance.

Replacing the sum by a product over the eigenvalues we finally find:

$$\mathcal{Z}_k = \sum_{x_0} \frac{1}{\mathrm{Sdet}\hat{\mathcal{L}}_{x_0}} = \sum_{\{Y_\lambda\}} \prod_{\lambda,\tilde{\lambda}}^{N} \prod_{s \in Y_\lambda} \frac{1}{E(s)(E(s) - \epsilon)} \tag{3.16}$$

with

$$E(s) = a_{\lambda\tilde{\lambda}} - \epsilon_1 h(s) + \epsilon_2 (v(s) + 1) \,. \tag{3.17}$$

3.2 $\mathcal{N} = 2$ supersymmetric theories with fundamental matter

In the presence of N_F fundamental hypermultiplets, the multi-instanton action gets a new contribution which can be written as [11]

$$S_{hyp} = -Q_\epsilon \,\mathrm{Tr} \left[h_f^\dagger \mathcal{K}_f + \mathcal{K}_f^\dagger h_f \right] , \tag{3.18}$$

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where $(\mathcal{K}_f, h_f)$, $f = 1, \ldots, N_F$ represent respectively the fermionic collective coordinates for the matter fields and their bosonic auxiliary variables. They are all matrices transforming in the $(\bar{k}, N_F)$ representation of the $\mathrm{U}(k) \times \mathrm{U}(N_F)$ group.[6] From the brane-engineering point of view, $(\mathcal{K}_f, h_f)$ are the massless excitation modes of open strings stretching between k D(-1) and N_F D7 fractional branes.

The BRST transformations of these fields are given by

$$\begin{aligned} Q_\epsilon \mathcal{K}_f &= h_f \qquad Q_\epsilon h_f = \phi \mathcal{K}_f + m_f \mathcal{K}_f \,, \\ Q_\epsilon \mathcal{K}_f^\dagger &= h_f^\dagger \qquad Q_\epsilon h_f^\dagger = -\mathcal{K}_f^\dagger \phi - m_f \mathcal{K}_f^\dagger \,, \end{aligned} \tag{3.19}$$

m_f being the mass of the f-th flavour. The vector field Q^* generating the above Q_ϵ action in the supermoduli space is given by

$$Q^* = h_f \frac{\partial}{\partial \mathcal{K}_f} + (\phi + m_f)\mathcal{K}_f \frac{\partial}{\partial h_f} \,, \tag{3.20}$$

which has to be added to the vector field (3.5) for the pure $\mathcal{N} = 2$ theory. The critical points are still given by (3.6), since the new components (3.20) of the vector field are set to zero simply by imposing $h_f = 0$. From (3.20) it follows that the contribution of each flavour f to the supertrace is simply

$$\delta\chi = -T_{m_f} \times V = -\sum_{\lambda}^{N} \sum_{s \in Y_\lambda} T_{a_\lambda} T_1^{-j_\lambda+1} T_2^{-i_\lambda+1} T_{m_f} \,, \tag{3.21}$$

with $T_{m_f} = e^{i m_f}$ an element of the maximal torus $\mathrm{U}(1)^{N_F} \subset \mathrm{U}(N_F)$. Taking into account the contribution of the N_F hypermultiplets, (3.16) becomes then

$$\mathcal{Z}_k = \sum_{\{Y_\lambda\}} \prod_{\lambda, \tilde{\lambda}}^{N} \prod_{s \in Y_\lambda} \frac{F(s)}{E(s)(E(s) - \epsilon)} \,, \tag{3.22}$$

where we defined

$$F(s) = \prod_{f=1}^{N_F} (\varphi(s) + m_f) \,, \tag{3.23}$$

with $\varphi(s) = \varphi_{i_\lambda j_\lambda}$ given by (3.7).

3.3 $\mathcal{N} = 2^*$ supersymmetric theories with gauge group $\mathrm{SU}(N)$

Our techniques in the previous subsections can be straightforwardly extended to the $\mathcal{N} = 2^*$ case. As we mentioned before, one can identify the parameter m in $\mathrm{U}(1)_m$ with the mass of the $\mathcal{N} = 2$ hypermultiplet. Notice that this identification was already implicit in our $\mathcal{N} = 2$ analysis above since the fields projected out in the reduction $\mathcal{N} = 4 \to \mathcal{N} = 2$ are precisely those charged under $\mathrm{U}(1)_m$.

[6] We explicit the flavour index f for convenience.

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The action of the BRST operatore, Q^*, is now given by:

$$
\begin{aligned}
Q^* \;&= \mu_I \frac{\partial}{\partial I} + \mu_J \frac{\partial}{\partial J} + K \frac{\partial}{\partial \mu_K} + L \frac{\partial}{\partial \mu_L} + \mathcal{M}_\ell \frac{\partial}{\partial B_\ell} + H_v \frac{\partial}{\partial \chi_v} + \eta \frac{\partial}{\partial \bar{\phi}} + (\phi - a) I \frac{\partial}{\partial \mu_I} + \\
&+ (-\phi + a + \epsilon) J \frac{\partial}{\partial \mu_J} + (\phi - a - m)\mu_K \frac{\partial}{\partial K} + (-\phi + a + \epsilon - m)\mu_L \frac{\partial}{\partial L} + \\
&+ ([\phi, B_\ell] + \lambda_\ell B_\ell) \frac{\partial}{\partial \mathcal{M}_\ell} + ([\phi, \chi_v] + \lambda_v \chi_v) \frac{\partial}{\partial H_v} + [\phi, \bar{\phi}] \frac{\partial}{\partial \eta} \,. \qquad (3.24)
\end{aligned}
$$

The critical points are again given by (3.7) since for generic m the condition $Q^* = 0$ requires $B_3 = B_4 = H_v = 0$. Reading from (3.24) the spectrum of eigenvalues we find:

$$
\chi = \left(1 - T_m^{-1}\right) \left[V^* \times V \times (T_1 + T_2 - T_1 T_2 - 1) + W^* \times V + V^* \times W \times T_1 T_2 \right] \qquad (3.25)
$$

Remarkably the contributions of massive fields match that of the $\mathcal{N} = 2$ in (3.12) but with eigenvalues shifted by $-m$. The final result can then be written as[7]

$$
\mathcal{Z}_k = \int \prod_{I=1}^{k} d\varphi_I \frac{\prod_{I<J} \varphi_{IJ}^2}{\mathrm{Sdet}\,\mathcal{L}} = \sum_{x_0} \frac{1}{\mathrm{Sdet}\hat{\mathcal{L}}_{x_0}} = \sum_{\{Y_\lambda\}} \prod_{\lambda, \tilde{\lambda}=1}^{N} \prod_{s \in Y_\lambda} \frac{(E(s) - m)(E(s) - \epsilon + m)}{E(s)(E(s) - \epsilon)} \,. \qquad (3.26)
$$

Notice that now the superdeterminant reduce to a product over $2kN$ bosonic and $2kN$ fermionic factors. Moreover, as in the $\mathcal{N} = 2$ case, the superdeterminant is non-trivial due to the cancellation of zero eigenvalues between bosons and fermions.

3.3.1 Some explicit examples: $k = 1, 2$

Here, for the sake of completeness, we explicitly evaluate formula (3.26) for $k = 1, 2$, recovering the results in [14, 21]. It is useful to introduce the following definitions:

$$
\begin{aligned}
f(x) &= \frac{(x - m)(x + m - \epsilon)}{x(x - \epsilon)} \qquad & g(x) &= \frac{1}{x(x - \epsilon)} \\
T_\alpha(x) &= \prod_{\tilde{\alpha} \neq \alpha} f(a_{\alpha\tilde{\alpha}} + x) \qquad & S_\alpha(x) &= \prod_{\tilde{\alpha} \neq \alpha} g(a_{\alpha\tilde{\alpha}} + x) \,. \qquad (3.27)
\end{aligned}
$$

In terms of these definitions we can rewrite:

$$
\mathcal{Z}_k = \sum_{\{Y_\lambda\}} \prod_{\lambda, \tilde{\lambda}=1}^{N} \prod_{s \in Y_\lambda} f(E(s)) \,. \qquad (3.28)
$$

Let us start by considering the $k = 1$ case: $Y_\alpha = \Box$, $Y_{\beta \neq \alpha} = \{\emptyset\}$. From the above definitions we have $v(s) = h(s) = 0$ for $\tilde{\lambda} = \alpha$ while $v(s) = -1, h(s) = 0$ for $\tilde{\lambda} \neq \alpha$. Summing up over diagrams of this kind one finds

$$
Z_1 = \sum_\alpha f(\epsilon_2) T_\alpha(0) \qquad (3.29)
$$

For $k = 2$ we have three diagrams:

[7] Once again, if we would plug (3.11) in (3.26) one would recover (3.25) in [14].

I) $Y_\alpha = \Box$, $Y_\beta = \Box$, $Y_{\gamma\neq\alpha,\beta} = \{\emptyset\}$:

$$Z_2^I = \frac{1}{2}\sum_{\alpha\neq\beta} f(\epsilon_2)^2 f(a_{\alpha\beta}+\epsilon_2) f(a_{\beta\alpha}+\epsilon_2)\frac{T_\alpha(0)T_\beta(0)}{f(a_{\alpha\beta})f(a_{\beta\alpha})} \tag{3.30}$$

The contribution $T_\alpha(0)/f(a_{\alpha\beta})$ comes from the product in (3.26) with $\lambda = \alpha$, $\tilde\lambda \neq \alpha, \beta$ for which $h(s) = 0, v(s) = -1$. The term $\lambda, \tilde\lambda = \alpha, \beta$, i.e. $h(s) = v(s) = 0$ gives $f(\epsilon_2)$ or $f(a_{\alpha\beta} + \epsilon_2)$ in the case of $\lambda = \tilde\lambda = \alpha$ and $\lambda = \alpha$, $\tilde\lambda = \beta$ respectively. Similar contributions come from terms with $\alpha \leftrightarrow \beta$ exchanged.

II) $Y_\alpha = \Box\Box$, $Y_{\tilde\alpha\neq\alpha} = \{\emptyset\}$:

$$Z^{II} = \sum_\alpha f(\epsilon_2) f(\epsilon_2 - \epsilon_1) T_\alpha(0) T_\alpha(-\epsilon_1)\,. \tag{3.31}$$

Now $f(\epsilon_2)f(\epsilon_2 - \epsilon_1)$ comes from the terms in (3.26) with $\lambda = \tilde\lambda = \alpha$ i.e. $v(s) = 0, h(s) = 0, 1$, while the product over $\tilde\lambda \neq \lambda = \alpha$, $v(s) = -1$, $h(s) = 0, 1$ brings the T_α contributions.

Finally the third diagram is the transposition of the one above and its contribution can be read from (3.31) by exchanging $\epsilon_1 \leftrightarrow \epsilon_2$. Setting $\epsilon_1 = -\epsilon_2 = \hbar$ and using the identification [18, 14]:

$$Z(a, \epsilon_1, \epsilon_2) = \sum_k Z_k q^k = \exp\left(\frac{\mathcal{F}^{\text{inst}}}{\epsilon_1\epsilon_2}\right). \tag{3.32}$$

one recovers the results in [33]:

$$\begin{aligned}
\mathcal{F}_1 &= -\lim_{\hbar\to 0} \hbar^2 Z_1 = m^2 \sum_\alpha T_\alpha \\
\mathcal{F}_2 &= -\lim_{\hbar\to 0} \hbar^2 \left(Z_2 - \frac{1}{2}Z_1^2\right) = \sum_\alpha \left(\frac{1}{4}m^4 T_\alpha T_\alpha'' - \frac{3}{2}m^2 T_\alpha^2\right) + \\
&\quad + m^4 \sum_{\alpha\neq\beta} T_\alpha T_\beta \left(\frac{1}{a_{\alpha\beta}^2} - \frac{1}{2(a_{\alpha\beta}-m)^2} - \frac{1}{2(a_{\alpha\beta}+m)^2}\right)
\end{aligned} \tag{3.33}$$

with $T_\alpha = T_\alpha(0)$. The $\mathcal{N} = 2$ analog of formulae (3.29), (3.31) can be simply obtained by replacing $f(x), T_\alpha(x)$ by $g(x), S_\alpha(x)$ respectively given by (3.27). For SU(2), formulae obtained in this way match those of [20].

3.4 $\mathcal{N} = 4$ supersymmetric theories with gauge group SU(N)

This case can be easily deduced from (3.26) by taking the limit $m \to 0$. This limit gives $\hat{\mathcal{L}}_{x_0} = 1$, thus applying (2.14) we get

$$\mathcal{Z}_k = \int_M e^{S^{N=4}} = \sum_{\{k\}} 1 \tag{3.34}$$

with $\{k\}$ the partitions of k. That is the partition function of $\mathcal{N} = 4$ is the sum over all critical points of the vector field and it gives the Euler characteristic of the moduli space [22]. This correponds to the N-colored number of partitions of an integer k. Going now to the generating function we see that

$$\mathcal{Z} = \sum_{k\geq 0} Z_k q^k = \sum_{k\geq 0} q^k \sum_{\{k_\lambda\}} 1 = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^N}\,. \tag{3.35}$$

Defining

$$\mathcal{F} = \sum_{k>0} q^k \mathcal{F}_k = \ln \mathcal{Z} = N \sum_{n>0} \ln(1-q^n) = N \sum_{k>0} q^k \sum_{d|k} \frac{1}{d}\,, \tag{3.36}$$

one finds [8] $\mathcal{F}_k = N \sum_{d|k} 1/d$. This result was already announced in [10] and motivated in [35] on the basis of a reasoning coming from string theory: in [26] the effective action of a single D3 brane of the IIB theory at order α'^4 was computed. The coupling is given by the modular invariant function $h(\tau,\bar{\tau}) = \ln|\tau_2 \eta(\tau)^4|$. By computing the generating functional of the instanton induced contributions to the scattering amplitude on the D3 brane and comparing with the results of [26], the value of $\mathcal{F}_k$ is found. Here the result is recovered by a direct evaluation of the instanton contributions.

The fact that the result of (3.35) is a function with particular properties under modular transformations is a very satisfying feature. Multi-instanton calculus exactly reproduces the important features of mathematical objects which have been studied with very different techniques.

Acknowledgments

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A. The momentup map and the Duistermaat-Heckman formula

Assume that M is symplectic, with symplectic form ω, that G acts on M by symplectomorphisms (i.e., $g^*\omega = \omega$ for all $g \in G$), and that this action admits a momentum map $\mu\colon M \to g^*$. Let $\alpha = \mu + \omega$. Here $deg(\alpha) = 2$, since ω is a two-form and μ is a linear functional on $\mathfrak{g}$, see (2.7). Now,

$$(g\cdot\alpha)(\xi) = g^*(\mu(ad_{\mathfrak{g}^{-1}}\xi)) + g^*\omega = \mu(\xi) + \omega = \alpha(\xi) \tag{A.1}$$

[8] A similar computation for the moduli space of instantons of winding number $k = 1/2$ on a Eguchi-Hanson manifold was carried out in [34]

since g acts as a symplectomorphism and μ is a momentum map. By definition, ω is closed and μ is a function, so that

$$(D\alpha)(\xi) = (d\alpha(\xi) - i_{\xi^*}\alpha(\xi)) = (d(\mu+\omega) - i_{\xi^*}(\mu+\omega)) = d\mu(\xi) - i_{\xi^*}\omega = 0\,. \qquad \text{(A.2)}$$

It then follows that $\alpha = \mu + \omega$ is an equivariantly closed form and the conditions of the localization formula are met. Plugging in (2.14) we get the Duistermaat-Heckman formula

$$\int_X e^{\mu+\omega} = \int_X \frac{\omega^{n/2}}{(n/2)!} e^{\mu} = (-2\pi)^{n/2} \sum_{x_0} \frac{\alpha_0(\xi)(x_0)}{\det^{\frac{1}{2}} \mathcal{L}_{x_0}}\,. \qquad \text{(A.3)}$$

since $\alpha_0(\xi) = e^{\mu}$.

Let now specify to our case where $G = \mathrm{U}(k)\times\mathrm{U}(1)^{N-1}\times T^2$. The condition of vanishing potential allows to take the Cartan part of the $\mathrm{U}(N)$ algebra in G. Given the symplectic form

$$\omega = dB_1 \wedge dB_1^\dagger + dB_2 \wedge dB_2^\dagger + dI \wedge dI^\dagger - dJ^\dagger \wedge dJ = dx \wedge dx^\dagger \qquad \text{(A.4)}$$

and the component of the vector field

$$(Q^*)^i = (\phi I - Ia, -J\phi + aJ + \epsilon J, [\phi, B_1] + \epsilon_1 B_1, [\phi, B_2] + \epsilon_2 B_2) \qquad \text{(A.5)}$$

we compute $d\mu = i_{Q^*}\omega = (Q^*)^i (dx^\dagger)^i$ from which

$$\mu = ([\phi, B_1] + \epsilon_1 B_1)B_1^\dagger + ([\phi, B_2] + \epsilon_2 B_2)B_2^\dagger + (\phi I - Ia)I^\dagger + J^\dagger(-J\phi + aJ + \epsilon J)\,. \quad \text{(A.6)}$$

How come that localization formulae with "actions" μ and $S^{\mathcal{N}=2}$ give the same results? The answer lies in $\mathcal{L}_{x_0}$. To determine its eigenvalues the only information we need is to know the components of the vector field Q^*. There is no reference to any action. All the information is encoded in the BRST transformations.

B. Multi-instanton actions: an overview

Here we collect some background material. Following [25] we decompose the quantum numbers of the D(-1)-D3 system in terms of the reduced euclidean Lorentz group $\mathrm{SO}(6)\times\mathrm{SO}(4)$. The ten dimensional spinor and gauge connection are taken as

$$\begin{aligned} \psi &= \sqrt{\frac{\pi}{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} \mathcal{M}^A_\beta \\ 0 \end{pmatrix} + \sqrt{\frac{\pi}{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ \bar\lambda^A_{\dot\beta} \end{pmatrix}, \\ A_M &= (\chi_a, a'_n)\,, \end{aligned}$$

while the low energy limit of the strings stretched between the k Dp and N D(p+4)-branes is given by the fields $(w_{\dot\alpha}, \mu^A; \bar w^{\dot\alpha}, \bar\mu^A)$. We have denoted by $a = 1,\ldots,6$ the indices of SO(6), by $A = 1,\ldots,4$ those of $\mathrm{SU}(4) \cong \mathrm{SO}(6)$ and by $\alpha, \dot\alpha = 1, 2$ those of $\mathrm{SO}(4) \cong \mathrm{SU}(2)\times\mathrm{SU}(2)$. The ADHM action of such system is given by [25]

$$S_{k,N} = \frac{1}{g_0^2} S_G + S_K + S_D \qquad \text{(B.1)}$$

with

$$
\begin{aligned}
S_G &= \mathrm{tr}_k \left(-[\chi_a, \chi_b]^2 + \sqrt{2} i\pi \lambda_{\dot\alpha A} \left[\chi^\dagger_{AB}, \bar\lambda^{\dot\alpha}_B \right] - D^c D^c \right) \\
S_K &= -\mathrm{tr}_k \left([\chi_a, a_n]^2 - \chi_a \bar w^{\dot\alpha} w_{\dot\alpha} \chi_a + \sqrt{2} i\pi \mathcal{M}'^{\ \alpha A} \left[\chi_{AB} \mathcal{M}'^{\ B}_\alpha \right] - 2\sqrt{2} i\pi \chi_{AB} \bar\mu^A \mu^B \right) \\
S_D &= \mathrm{tr}_k \left(i\pi \left(- \left[a_{\alpha\dot\alpha}, \mathcal{M}'^{\ \alpha A} \right] + \bar\mu^A w_{\dot\alpha} + \bar w_{\dot\alpha} \mu^A \right) \bar\lambda^{\dot\alpha}_A + D^c \left(\bar w \tau^c w - i \bar\eta^c_{mn} [a_m, a_n] \right) \right) \quad \text{(B.2)}
\end{aligned}
$$

To go to an action with a lower number of supersymmetries it is sufficient to repeat the above construction in the case of fractional branes. A fractional brane lives at orbifold singularities and its low energy field theory must be invariant under the action of the discrete group by which we mod the original space-time. The set of invariant fields is clearly smaller than the original one and the final theory has thus less supersymmetries [11]. To be consistent with the notation adopted for the ADHM variables in the second section, we now set[9]

$$
\begin{aligned}
w_{\dot\alpha} &= \begin{pmatrix} I^\dagger \\ J \end{pmatrix} , \\
B_1 &= -a'_0 + i a'_3 \qquad B_2 = -a'_2 + i a'_1 , && \text{(B.3)} \\
B_3 &= \frac{1}{\sqrt{2}} (-\chi_1 + i\chi_4) \qquad B_4 = \frac{1}{\sqrt{2}} (-\chi_2 + i\chi_5) . && \text{(B.4)} \\
\phi &= \frac{1}{\sqrt{2}} (-\chi_3 + i\chi_6) \qquad \bar\phi = \frac{1}{\sqrt{2}} (-\chi_3 - i\chi_6) . && \text{(B.5)}
\end{aligned}
$$

Let's discuss the $\mathcal{N} = 2^*$ case: the fields in (B.4), together with the fermionic components given by $A = 3, 4$ and some new auxiliary fields (K, L, H_2, H_3) give rise to the massive hypermultiplets with bosonic components $(K, L, B_3, B_4, H_2, H_3)$ and fermionic components $(\mu^3, \mu^4, \bar\lambda^{\dot\alpha}_{3,4}, \mathcal{M}'^{\ 3,4}_\alpha)$. By renaming $\mu^{3,4} \to \mu_{K,L}$, $\mathcal{M}'^{\ 3,4}_\alpha \to \chi_{2,3}$ and $\bar\lambda^{\dot\alpha}_{3,4} \to \mathcal{M}_{3,4}$ we finally get the fields entering the action (2.3) and transforming as (2.4) with $a = \epsilon = 0$. Notice that upon the rescalings $(I, J^\dagger, B_1, B_2) \to g_0^{1/2} (I, J^\dagger, B_1, B_2)$ and $(B_3, B_4, \phi) \to g_0^{-1/2} (B_3, B_4, \phi)$ and integration on the auxiliary fields $(\vec\chi, \vec H)$, the action (2.3) reproduces (B.1), integrated with respect to $(\bar\lambda^{\dot\alpha}_A, D^c)$.

We now specialize the above discussion to the case of fractional branes. After the $\mathbb{Z}_2$ projection the multi-instanton action can be read from (B.2) with fermionic indices A, B now restricted to $\dot A, \dot B = 1, 2$ (in the fundamental of the automorphism group) which corresponds to set the entire massive hypermultiplet to zero see (3.1). The action thus obtained can be seen as the implementation à la BRST of the ADHM constraints, which we "twist" by identifying $\dot A$ with $\dot\alpha$. The constraints are now given by (3.3). To them we associate the doublet of auxiliary fields $(\bar\lambda^A_{\dot\alpha}, D^c)$ in (B.2) which we rename $(\chi = \chi_{\mathbb{R}}, \chi_{\mathbb{C}}; H = H_{\mathbb{R}}, H_{\mathbb{C}})$ and the doublet $(\bar\phi, \eta)$. Given all this, after introducing a v.e.v. for the scalar field we get the action of $\mathcal{N} = 2$ SYM [11]

$$
\begin{aligned}
S^{N=2} = Q\, \mathrm{Tr} \Big\{ & \left[\mu_I \left(I^\dagger \bar\phi - \bar a I^\dagger \right) + \mu_J \left(\bar\phi J^\dagger - J^\dagger \bar a \right) + \mathcal{M}_{\hat\ell} \left[\bar\phi, B^\dagger_{\hat\ell} \right] \right] + \text{h.c.} + \\
& + \chi_{\mathbb{R}} \mathcal{E}_{\mathbb{R}} + \chi_{\mathbb{C}} \mathcal{E}_{\mathbb{C}} + \frac{1}{g_0^2} \left(\eta \left[\phi, \bar\phi \right] + \chi \cdot H \right) \Big\} , \qquad \text{(B.6)}
\end{aligned}
$$

invariant under the BRST transformations (3.4) with $\epsilon_\ell = 0$.

[9] We choose $\sigma^n_{\alpha\dot\alpha} = (-1, i\tau^c)$.

We are now ready to discuss the properties of (B.6). For the sake of clarity let us, for the moment, disregard the auxiliary fields implementing the constraints by setting $H_{\mathbb{R}}, H_{\mathbb{C}}$ and their fermionic partners to zero. Moreover notice that the transformations (3.4) have been improperly called BRST, since they do not square to zero. The complete BRST operator on the ADHM space has been constructed in [8] and reads

$$\begin{aligned}
sI &= \mu_I - CI \qquad s\mu_I = \phi I - Ia - C\mu_I \\
sJ &= \mu_J + JC \qquad s\mu_J = -J\phi + aJ + \mu_J C\,, \\
sB_1 &= \mathcal{M}_1 - [C, B_1] \qquad s\mathcal{M}_1 = [\phi, B_1] - [C, \mathcal{M}_1] \\
sB_2 &= \mathcal{M}_2 - [C, B_2] \qquad s\mathcal{M}_2 = [\phi, B_2] - [C, \mathcal{M}_2]\,, \\
s\phi &= -[C\phi] \qquad sC = (\phi - a) - [C, C]
\end{aligned} \tag{B.7}$$

where C is a $\mathrm{U}(k)$ connection acting on the fields as

$$C \cdot (I, J, B_1, B_2) = (CI, -JC, [C, B_1], [C, B_2])\,. \tag{B.8}$$

The Q operator correspond to the covariant derivative on the ADHM moduli space $Q = s + C\cdot$. In terms of the BRST operator s, the action (B.6) can be written as

$$S^{N=2} = s\,\mathrm{Tr}(\mu_I \bar{a} I^\dagger + J^\dagger \bar{a} \mu_J + \text{h.c.}) = s\ \alpha = Q\alpha\,. \tag{B.9}$$

According to the grading (2.7), α is a 3-form and since it is $\mathrm{U}(k) \times \mathrm{U}(N)$ equivariant (see (4.39) in [8]) the actions of s and Q on it give the same result. By substituting the fermions with their expressions (B.7), α becomes a bosonic form on the moduli space of instantons as suggested in [8] and since $s^2 = 0$, the BRST operator can be interpreted as a *bona fide* derivative. In [20] it is suggested that, since the action of s and Q on α are the same, it can be more convenient to interpret Q as the equivariant derivative D introduced earlier in (2.9) and drop the connection C. This does not mean to drop C altogether but only when it acts on the form α. In [8] it is, in fact, shown that the presence of C is crucial to recover the correct measure on the instanton moduli space.

It is immediate to see that, due to BRST invariance, α is equivariantly closed and that the infinitesimal action of the bosonic vector field ξ^* can be read from the action of Q^2 (2.12) which is the Lie derivative. The localization theorems could now be applied. The bosonic part of the action (the part of the action which is a zero form with respect to differentials of the ADHM variables) is given by $i_{\xi^*}\alpha$ which is positive semi-definite. Then its zeroes are the critical points which could also be obtained by computing the Q^2 on the bosonic variables

$$\begin{aligned}
Q^2 I &= \phi I - Ia \\
Q^2 J &= -J\phi + aJ \\
Q^2 B_l &= [\phi, B_l]\,.
\end{aligned} \tag{B.10}$$

These are exactly the critical points found in [13]. As in that paper these critical points are rather critical surfaces and the application of the localization theorem is rather cumbersome. The useful suggestion now comes from [16, 15, 14]: we can introduce a further symmetry in the problem which, without changing the cohomology, can "reduce" the critical surfaces to isolated critical points. It acts on the coordinate of spacetime as two independent rotations in the x_1, x_2 and x_3, x_4 planes. The group element describing such rotations is $T^2 = (t_1, t_2)$ with $t_i = \exp i\epsilon_i$, $i = 1, 2$ acting on the complex coordinates as $(z_1, z_2) \to (t_1 z_1, t_2 z_2)$. There is clearly a wide margin of arbitrarity in the choice of the parametrization of the T^2, since we can always arbitrarily rescale the complex coordinates. The only condition is to leave (3.3) invariant. Our action and BRST transformation must be changed accordingly to accomodate for the new symmetry. The resulting action and BRST transformations are described in the main body of the paper.

C. Higher rank Hilbert-Chow morphisms

In this section we will comment on Proposition 5.8 in [12] which gives the character, $T_Z(\mathbb{C}^2)^{[n]}$, at the fixed points, Z, of the tangent space to the Hilbert scheme $(\mathbb{C}^2)^{[n]}$ of n points on $\mathbb{C}^2$. With a little generalization, from this formula one can extract the eigenvalues of $Sdet$ in (3.16), (3.26). Here we will try to "translate" the setting of [12] in the language we have used for this paper.

From the definition of the map $\mathcal{L}_{x_0}$ it should become clear why we are after the character (or eigenvalues) of the tangent space.

Now consider the problem, given a self-dual field strength $F_{\mu\nu}$ and a vector potential A_μ of investigating the infinitesimal variations δA_μ preserving the self-duality of $F_{\mu\nu}$. Then

$$(d_2 \delta A)_{\mu\nu} = \Pi_{\mu\nu}{}^{\alpha\beta}(D_\alpha \delta A_\beta - D_\beta \delta A_\alpha) = 0. \tag{C.1}$$

D_α is the gauge covariant derivative and $\Pi_{\mu\nu}{}^{\alpha\beta}$ the projector on the anti-self-dual part of a tensor. Among the solutions of (C.1), there are those arising from infinitesimal gauge transformations, ε, of the gauge field. They are of the form

$$(d_1 \varepsilon)_\mu = D_\mu \varepsilon\,. \tag{C.2}$$

Two solutions of (C.1) are gauge equivalent if they differ by a field of the form (C.2). The problem of finding the number of gauge inequivalent solutions to (C.1) is more conveniently treated by representing the tangent space $T_Z(\mathbb{C}^2)^{[n]}$ as the quotient $\mathrm{Ker}\, d_2 / \mathrm{Im}\, d_1$ associated to the complex

$$\mathrm{Hom}(V,V) \xrightarrow{d_1} \begin{array}{c} \mathrm{Hom}(V, Q\otimes V) \\ \oplus \\ \mathrm{Hom}(W,V) \\ \oplus \\ \mathrm{Hom}(V, \bigwedge^2 Q \otimes W) \end{array} \xrightarrow{d_2} \mathrm{Hom}(V,V)\otimes \bigwedge{}^2 Q \tag{C.3}$$

introduced to prove Proposition 5.8 in [12]. In (C.3) the symmetry with respect to the action of the two torus $T^2 = (t_1, t_2)$ has been taken into account by introducing the doublet Q. $\bigwedge^2 Q = t_1 t_2 = \exp\{i\epsilon\}$ is the totally antisymmetric combination which that is the determinant. The correspondence between this notation and that of the rest of the paper is

$$\begin{aligned}
B_l &: \ \mathrm{Hom}(V, Q \otimes V) \\
I^\dagger &: \ \mathrm{Hom}(W, V) \\
J^\dagger &: \ \mathrm{Hom}\left(V, \bigwedge{}^2 Q \otimes W\right) \\
\chi_{\mathbb{R}} &: \ \mathrm{Hom}(V, V) \\
\chi_{\mathbb{C}} &: \ \mathrm{Hom}(V, V) \otimes \bigwedge{}^2 Q\,.
\end{aligned} \tag{C.4}$$

Then

$$\begin{aligned}
T_Z = {} & \mathrm{Hom}(V, Q \otimes V) + \mathrm{Hom}(W, V) + \mathrm{Hom}\left(V, \bigwedge{}^2 Q \otimes W\right) - \\
& - \mathrm{Hom}(V, V) - \mathrm{Hom}(V, V) \otimes \bigwedge{}^2 Q = V^* \otimes V \otimes \left(Q - \bigwedge{}^2 Q \otimes W - 1\right) + \\
& + W^* \otimes V + V^* \otimes W \otimes \bigwedge{}^2 Q\,.
\end{aligned} \tag{C.5}$$

At the critical point the tangent space can be decomposed in terms of the quantum numbers of $T^2 \times \mathrm{U}(1)^{n-1}$ giving (3.13).

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